

Supersymmetry and the Odd Poisson Bracket

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Some applications of the odd Poisson bracket developed by Kharkov's theorists are represented.

1. INTRODUCTION

The theory of supersymmetry substantially expanded our ideas not only on the possible types of symmetry relations but also on the way of the dynamics description. Indeed, with the discovery of supersymmetry [1–4], which extends to a superspace the configuration space x by adding to it anticommuting (Grassmann) coordinates θ and extends by this also the phase space with the coordinates x and p , the possibility arises to generalize directly the Poisson bracket to the even Poisson bracket by adding the Martin bracket [5] which canonically conjugates the Grassmann variables θ with each other. However, mathematicians (first of them was Buttin [6]) proved that this is not the only possibility of generalization of the Poisson bracket to the case of the phase superspace having both the even and odd (with respect to the Grassmann parity) variables. In addition to the even Poisson–Martin bracket indicated above there exists another possibility to define on the phase superspace the bracket operation of the Poisson type which canonically conjugates phase variables of the opposite Grassmann parity. These two brackets confine all possible cases of combinations of the Grassmann parity of canonically conjugated quantities in the superspace. The essential difference of the latter bracket from the even bracket is the existence in it of a nontrivial Grassmann parity equal to unity, and because of this it is called the odd Poisson bracket or simply an odd bracket (OB).

The even Poisson and Martin brackets are well known. Their quantization leads to respectively two forms of quantum statistics: the Bose–Einstein and the Fermi–Dirac. The study and physical use of the OB has started relatively recent. In physics the OB has firstly appeared as an adequate language for the quantization of the gauge theories in the well-known Batalin–Vilkovisky scheme [7]. Leites [8] assumed that the OB is connected with another type of classical mechanics. Manin and Radul have used the OB for the description of the supersymmetric integrable systems [9] and Kupershmidt has applied it for the description of the one-dimensional ideal liquid with fermionic degrees of freedom [10].

However, the apparent dynamical role of the OB was not understood quite well till papers [11,12]. In [11] a possibility of the reformulation of Hamiltonian dynamics on the basis of the OB with the use of the dynamically equivalent Grassmann–odd Hamiltonian was proved for the classical supersymmetric Witten mechanics having two even q , p and two odd θ_1 , θ_2 phase variables and in [12] this result was generalized for the Hamilton systems with an arbitrary equal number of pairs of even and odd (relative to the Grassmann grading) phase coordinates. Nevertheless, there are such Hamiltonian systems which have a description only by means of the OB and can not be reformulated in terms of the even bracket. In particular, as it was shown in [13], usual hydrodynamics is just such a system with three even and three odd phase coordinates. Hamiltonian formulation of dynamics in the OB is closely related with Lagrangian formulation hav-

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ing a Grassmann-odd Lagrangian. As an example of this in [14] an action with the Grassmann-odd Lagrangian for the supersymmetric classical Witten mechanics has been constructed.

The direct approach to the quantization of the OB leads to the odd Planck constant which has no physical interpretation. Therefore in [15] another, free from this difficulty, prescription for the canonical quantization of the OB was suggested and several odd-bracket quantum representations for the canonical variables were also obtained. In contrast with the even bracket case, some of the odd-bracket quantum representations turn out to be nonequivalent [16,17]. The direct connection of the odd-bracket quantum representations with the quantization of classical Hamiltonian dynamics based on the OB has been demonstrated on an example of the one-dimensional supersymmetric oscillator [17]. Another application of the odd-bracket quantum representations is connected with the possibility of realization of the idea of the composite (spinor) structure of coordinates, which illustrated on the examples of superspaces of the orthosymplectic supergroups $OSp(N, 2k)$ for $k = 1, 2$ [15,18].

Recently a linear OB on Grassmann algebra has been introduced [19]. It was revealed that with the OB, which corresponds to a semi-simple Lie group, both a Grassmann-odd Casimir function and invariant (relative to this group) nilpotent differential operators of the first, second and third order with respect to Grassmann derivatives are naturally related and enter into a finite-dimensional Lie superalgebra. A connection of the quantities, forming this superalgebra, with the BRST charge, Δ -operator and ghost number operator was indicated.

2. CLASSICAL ASPECT OF THE ODD BRACKET

2.1. Properties of the odd Poisson bracket

First, we recall the necessary properties of various graded Poisson brackets. The even and odd brackets in terms of the real even $y_i = (q^a, p_a)$ and odd $\eta^i = \theta^\alpha$ canonical variables have, re-

spectively, the form

$$\{A, B\}_o = A \left[\sum_{a=1}^n \left(\overleftarrow{\partial}_{q^a} \overrightarrow{\partial}_{p_a} - \overleftarrow{\partial}_{p_a} \overrightarrow{\partial}_{q^a} \right) - i \sum_{\alpha=1}^{2n} \overleftarrow{\partial}_{\theta^\alpha} \overrightarrow{\partial}_{\theta^\alpha} \right] B ; \quad (1)$$

$$\{A, B\}_1 = A \sum_{i=1}^N \left(\overleftarrow{\partial}_{y_i} \overrightarrow{\partial}_{\eta^i} - \overleftarrow{\partial}_{\eta^i} \overrightarrow{\partial}_{y_i} \right) B , \quad (2)$$

where $\overleftarrow{\partial}$ and $\overrightarrow{\partial}$ are the right and left derivatives, and the notation $\partial_x = \frac{\partial}{\partial x}$ is introduced. By introducing apart from the Grassmann grading $g(A)$ of any quantity A its corresponding bracket grading $g_\epsilon(A) = g(A) + \epsilon \pmod{2}$ ($\epsilon = 0, 1$), the grading and symmetry properties, the Jacobi identities and the Leibnitz rule are uniformly expressed for the both brackets (1,2) as

$$g_\epsilon(\{A, B\}_\epsilon) = g_\epsilon(A) + g_\epsilon(B) \pmod{2} , \quad (3)$$

$$\{A, B\}_\epsilon = -(-1)^{g_\epsilon(A)g_\epsilon(B)} \{B, A\}_\epsilon , \quad (4)$$

$$\sum_{(ABC)} (-1)^{g_\epsilon(A)g_\epsilon(C)} \{A, \{B, C\}_\epsilon\}_\epsilon = 0 , \quad (5)$$

$$\{A, BC\}_\epsilon = \{A, B\}_\epsilon C + (-1)^{g_\epsilon(A)g_\epsilon(B)} B \{A, C\}_\epsilon , \quad (6)$$

where (3)–(5) have the shape of the Lie superalgebra relations in their canonical form [20] with $g_\epsilon(A)$ being the canonical grading for the corresponding bracket.

In terms of arbitrary real dynamical variables $x^M = (x^m, x^\alpha) = x^M(y, \eta)$ with the same number of Grassmann even x^m and odd x^α coordinates the odd bracket (2) takes the form

$$\{A, B\}_1 = A \overleftarrow{\partial}_M \bar{\omega}^{MN}(x) \overrightarrow{\partial}_N B . \quad (7)$$

The matrix $\bar{\omega}_{MN}$, inverse to $\bar{\omega}^{MN}$

$$\bar{\omega}_{MN} \bar{\omega}^{NL} = \delta_M^L , \quad (8)$$

and consisting of the coefficients of the odd closed 2-form, in view of the odd bracket properties (3)–(5) can be represented in the form of the grading strength

$$\bar{\omega}_{MN} = \partial_M \bar{\mathcal{A}}_N - (-1)^{g(M)g(N)} \partial_N \bar{\mathcal{A}}_M , \quad (9)$$

where $g(M) = g(x^M)$ and $\partial_M = \partial/\partial x^M$. The coefficients of the 1-form $\bar{\mathcal{A}}(d) = dx^M \bar{\mathcal{A}}_M$ satisfy the conditions

$$g(\bar{\mathcal{A}}_M) = g(M) + 1, \quad (\bar{\mathcal{A}}_M)^+ = \bar{\mathcal{A}}_M. \quad (10)$$

As can be seen from (9) $\bar{\omega}_{MN}$ is invariant under gauge transformations

$$\bar{\mathcal{A}}'_M = \bar{\mathcal{A}}_M + \partial_M \bar{\chi} \quad (11)$$

with functions $\bar{\chi}$ as parameters.

2.2. Classical dynamics in terms of the odd Poisson bracket

Let us consider the Hamilton system containing an equal number n of pairs of even and odd with respect to the Grassmann grading real canonical variables. We require that the equations of motion of the system be reproduced both in the even Poisson-Martin bracket (1) with the help of the even Hamiltonian H and in the odd bracket (2) with the Grassmann-odd Hamiltonian \bar{H} , that is [11,12],

$$\frac{dx^M}{dt} = \{x^M, H\}_0 = \{x^M, \bar{H}\}_1, \quad (12)$$

where t is the proper time. Using definitions (7) and (1) together with (8), (9) the relations (12) can be represented as the equations

$$(\partial_M \bar{\mathcal{A}}_N - (-1)^{g(M)g(N)} \partial_N \bar{\mathcal{A}}_M) \omega^{NL} \partial_L H = \partial_M \bar{H} \quad (13)$$

to derive the unknown \bar{H} and $\bar{\mathcal{A}}_M$ under the given H and the even matrix ω^{MN} corresponding to the even bracket (1).

In order to solve equations (13) it is convenient to use such real canonical in the even bracket (1) coordinates x^M which contain among canonically conjugate pairs the pair consisting of the proper time t and the Hamiltonian H . It follows from (12) that the rest of the canonical quantities z^M would be the integrals of motion for the system considered: even $I_1, \dots, I_{2(n-1)}$ and odd $\Theta^1, \dots, \Theta^{2n}$. In terms of these coordinates x^M equations (13) take the form

$$(\partial_M \bar{\mathcal{A}}_t - \partial_t \bar{\mathcal{A}}_M) = \partial_M \bar{H}. \quad (14)$$

The quantities $\bar{\mathcal{A}}_M, \bar{\chi}$ and \bar{H} can be expanded in powers of the Grassmann variables Θ^α as

$$\bar{\mathcal{A}}_m = \sum_{k=1}^n \frac{i^{(k-1)(2k-1)}}{(2k-1)!} A_{m\alpha_1 \dots \alpha_{2k-1}} \Theta^{\alpha_1} \dots \Theta^{\alpha_{2k-1}}, \quad (15)$$

$$\bar{\mathcal{A}}_\alpha = \sum_{k=0}^n \frac{i^{k(2k+1)}}{(2k)!} B_{\alpha\alpha_1 \dots \alpha_{2k}} \Theta^{\alpha_1} \dots \Theta^{\alpha_{2k}}, \quad (16)$$

$$\bar{\chi} = \sum_{k=1}^n \frac{i^{(k-1)(2k-1)}}{(2k-1)!} \chi_{\alpha_1 \dots \alpha_{2k-1}} \Theta^{\alpha_1} \dots \Theta^{\alpha_{2k-1}}, \quad (17)$$

$$\bar{H} = \sum_{k=1}^n \frac{i^{(k-1)(2k-1)}}{(2k-1)!} h_{\alpha_1 \dots \alpha_{2k-1}} \Theta^{\alpha_1} \dots \Theta^{\alpha_{2k-1}}. \quad (18)$$

The Θ^α coefficients are the real Grassmann-even functions of the even variables $x^m = (t, H, I_1, \dots, I_{2(n-1)})$ and are chosen to be antisymmetric in the indices contracted with Θ^α . In terms of these functions the gauge transformations (11) have the form

$$\begin{aligned} \mathcal{A}'_{m\alpha_1 \dots \alpha_{2k-1}} &= A_{m\alpha_1 \dots \alpha_{2k-1}} + \\ \partial_m \chi_{\alpha_1 \dots \alpha_{2k-1}}, &\quad (k = 1, \dots, n); \end{aligned}$$

$$\begin{aligned} B'_{[\alpha\alpha_1 \dots \alpha_{2k}]} &= B_{[\alpha\alpha_1 \dots \alpha_{2k}]} + \\ \chi_{\alpha\alpha_1 \dots \alpha_{2k}}, &\quad (k = 0, 1, \dots, n-1); \end{aligned}$$

$$\begin{aligned} B'_{(\alpha\alpha_j)\alpha_1 \dots \alpha_{j-1}\alpha_{j+1} \dots \alpha_{2k}} &= \\ B_{(\alpha\alpha_j)\alpha_1 \dots \alpha_{j-1}\alpha_{j+1} \dots \alpha_{2k}}, &\quad \left(\begin{array}{l} k = 0, 1, \dots, n \\ j = 1, \dots, 2k \end{array} \right); \end{aligned} \quad (19)$$

where the expansion in the components with different symmetries of the indices has been used for the tensor antisymmetric in all indices but the first

$$\begin{aligned} B_{\alpha\alpha_1 \dots \alpha_{2k}} &= B_{[\alpha\alpha_1 \dots \alpha_{2k}]} + \\ \frac{2}{N+1} \sum_{j=1}^N (-1)^{j-1} B_{(\alpha\alpha_j)\alpha_1 \dots \alpha_{j-1}\alpha_{j+1} \dots \alpha_{2k}}. & \end{aligned}$$

The additive character of the transformations for the functions $B_{[\alpha\alpha_1 \dots \alpha_{2k}]}$ ($k = 0, 1, \dots, n-1$)

allows us to put them equal to zero in the expression (16) for $\bar{\mathcal{A}}_\alpha$ by choosing $\chi_{\alpha\alpha_1\dots\alpha_{2k}} = -B_{[\alpha\alpha_1\dots\alpha_{2k}]}$. This gauge choice amounts to the following gauge condition

$$\Theta^\alpha \bar{\mathcal{A}}_\alpha = 0 .$$

Using this condition and equations (14), we obtain the equality

$$\bar{H} = \bar{\mathcal{A}}_t . \quad (20)$$

which, being substituted again into (14), leads to the simple equations

$$\partial_t \bar{\mathcal{A}}_M = 0 . \quad (21)$$

Thus, in consequence of (21), the solution of equations (14) for $\bar{\mathcal{A}}_M$ and \bar{H} in the chosen gauge resides in that the nonzero coefficients $A_{m\alpha_1\dots\alpha_{2k-1}}$ and $B_{(\alpha\alpha_j)\alpha_1\dots\alpha_{j-1}\alpha_{j+1}\dots\alpha_{2k}}$ in expansions (15,16) for $\bar{\mathcal{A}}_M$ are the arbitrary functions (denoted as $a_{m\alpha_1\dots\alpha_{2k-1}}$ and $b_{(\alpha\alpha_j)\alpha_1\dots\alpha_{j-1}\alpha_{j+1}\dots\alpha_{2k}}$, respectively) of all, except the proper time t , even variables H and $I_1, \dots, I_{2(n-1)}$, and the odd Hamiltonian is expressed in terms of these functions with the help of equation (20).

Using the gauge transformations (19) with the arbitrary functions $\chi_{\alpha_1\dots\alpha_{2k-1}}(t, H, I)$, we obtain the general solution of equations (14) in the arbitrary gauge:

$$\begin{aligned} A_{m\alpha_1\dots\alpha_{2k-1}} &= a_{m\alpha_1\dots\alpha_{2k-1}}(H, I) + \\ &\quad \partial_m \chi_{\alpha_1\dots\alpha_{2k-1}}(t, H, I) ; \\ B_{\alpha\alpha_1\dots\alpha_{2k}} &= \\ \frac{2}{2k+1} \sum_{j=1}^{2k} (-1)^{j-1} b_{(\alpha\alpha_j)\alpha_1\dots\alpha_{j-1}\alpha_{j+1}\dots\alpha_{2k}}(H, I) + \\ &\quad \chi_{\alpha\alpha_1\dots\alpha_{2k}}(t, H, I) ; \end{aligned}$$

$$h_{\alpha_1\dots\alpha_{2k-1}} = a_{t\alpha_1\dots\alpha_{2k-1}}(H, I) .$$

Note that the solution of the analogous problem of finding the even brackets and the corresponding even Hamiltonians, which lead to the same equations of motion

$$\frac{dx^M}{dt} = \{x^M, H\}_0 = \{x^M, \tilde{H}\}_{\bar{0}} ,$$

has a similar structure but with the difference that the odd quantities $\bar{\mathcal{A}}_M, \bar{\chi}$ and \bar{H} has to be replaced by the even ones.

Thus, we extended the notion of the bi-Hamiltonian systems onto the case when the pairs of the Hamiltonian-bracket, giving the same equations of motion, have an opposite Grassmann grading.

2.3. Hydrodynamics description in the odd bracket

In [13] an approach for the description of hydrodynamics as a Hamilton system in terms of the odd bracket has been developed. This approach is a powerful tool for the description and construction of hydrodynamic invariants.

The velocity field for the hydrodynamic medium with coordinates $x^i(t)$ satisfies the equations

$$\frac{dx^i}{dt} = V^i(x, t) ,$$

which can be rewritten

$$\frac{dx^i}{dt} = \{x^i, H\}_1 = V^i(x, t)$$

by means of the odd bracket (2)

$$\{A, B\}_1 = A \sum_{i=1}^3 \left(\overleftarrow{\partial}_{x^i} \overrightarrow{\partial}_{\theta_i} - \overleftarrow{\partial}_{\theta_i} \overrightarrow{\partial}_{x^i} \right) B$$

with the help of the Grassmann-odd Hamiltonian

$$H = V^i(x, t) \theta_i .$$

For the concordance of the equations of motion for Grassmann coordinates θ_i

$$\frac{d\theta_i}{dt} = \{\theta_i, H\}_1 = -\partial_{x^i} V^k \theta_k \quad (22)$$

with the equations for the exterior differentials dx^i

$$\frac{d}{dt} dx^i = dx^k \partial_{x^k} V^i \quad (23)$$

let us introduce a Lie dragged metric tensor $g_{ik}(x, t)$

$$\partial_t g_{ik} + L_V g_{ik} = 0 ,$$

(L_V is the Lie derivative) which connects the field co-differentials θ_i with differentials dx^i

$$\theta_i = g_{ik} dx^k .$$

A super invariant I_s is a function of the super-space coordinates $z = (x^i, \theta_i)$ ($i = 1, 2, 3$)

$$I_s(z, t) = I + J^i \theta_i + S_i \frac{\epsilon^{ikl}}{\rho} \theta_k \theta_l + \frac{1}{\rho} \theta_1 \theta_2 \theta_3 \quad (24)$$

and obeys the equation for an invariant

$$\frac{dI_s}{dt} = \partial_t I_s + \{I_s, H\}_1 = 0 . \quad (25)$$

The θ_i coefficients of the super invariant (24) are well-known local hydrodynamic invariants: $I(x, t)$ is the Lagrangian invariant, $J^i(x, t)$ is a frozen-in field, $S_i(x, t)$ is a frozen-in surface and $\rho(x, t)$ is the invariant density. The θ_i components of the super equation (25) are the equations for the local invariants

$$\partial_t I + L_V I = 0 , \quad (26)$$

$$\partial_t J^i + L_V J^i = 0 , \quad (27)$$

$$\partial_t S_i + L_V S_i = 0 , \quad (28)$$

$$\partial_t \rho + \partial_{x^i} (V^i \rho) = 0 . \quad (29)$$

From two super invariants I_{s1} and I_{s2} we can construct new ones with the use of the usual multiplication

$$I_{s3} = I_{s1} I_{s2}$$

and with the help of the odd bracket composition

$$I_{s4} = \{I_{s1}, I_{s2}\} .$$

In consequence of the equations (22), (23) and (26)–(29) all terms of the super invariant (24) are separately conserved as well as the quantity

$$\frac{d}{dt} (\theta_i dx^i) = 0 . \quad (30)$$

Due to (30) we can pass from local invariants to integral ones by taking a Fourier transform of the super invariant (24)

$$\int I_s(z, t) \rho(x, t) \exp(\theta_i dx^i) d\theta^1 d\theta^2 d\theta^3 = I_{\text{int}} . \quad (31)$$

By performing in (31) the Berezin integration [20] over Grassmann variables θ_i we obtain as a result the ordinary integral conservation laws: "mass" conservation law, frozen-in field flux conservation law and conservation of the circulation **S**-invariant respectively

$$I_{\text{int}} = \int \rho I dx^1 \wedge dx^2 \wedge dx^3 + \int \rho \mathbf{J} d\mathbf{x} \wedge \mathbf{x} + \int S_i dx^i + 1$$

2.4. Dynamics with Grassmann-odd Lagrangian

Here we show that the idea stated in [12] about the possible existence of the dynamics formulation with the Grassmann-odd Lagrangian can be realized on the example of $d = 1, N = 2$ supersymmetric Witten's mechanics [21] in its classical version [11, 22].

Let us consider a system invariant with respect to the $N = 2$ ($\alpha = 1, 2$) supersymmetry of the proper time t

$$t' = t + i\epsilon^\alpha \theta^\alpha , \quad \theta'^\alpha = \theta^\alpha + \epsilon^\alpha ,$$

$$D_\alpha = \frac{\partial}{\partial \theta^\alpha} - i\delta_{\alpha\beta} \theta^\beta \frac{\partial}{\partial t} .$$

By using the covariant derivatives D_α and two real scalar superfields

$$\Phi(t, \theta^\alpha) = q(t) + i\psi_\alpha(t)\theta^\alpha + iF(t)\theta^\alpha\theta_\alpha ,$$

$$\Psi(t, \theta^\alpha) = \eta(t) + ia_\alpha(t)\theta^\alpha + i\Xi(t)\theta^\alpha\theta_\alpha ,$$

having the opposite values of the Grassmann grading g ($g(\Phi) = 0, g(\Psi) = 1$), the following supersymmetric action \bar{S} with the Grassmann-odd Lagrangian \bar{L} ($g(\bar{L}) = 1$) can be constructed

$$\begin{aligned} \bar{S} &= \int dt d\theta_2 d\theta_1 \left[-\frac{1}{2} D^\alpha \Psi D_\alpha \Phi + i\Psi W(\Phi) \right] \\ &= \int dt \bar{L} , \quad (32) \end{aligned}$$

where $W(\Phi)$ is an arbitrary real function of Φ . Excluding in (32) the auxiliary fields F and Ξ , we obtain

$$\begin{aligned} \bar{L} &= \dot{\eta} \dot{q} + \frac{i}{2} (a^\alpha \dot{\psi}^\alpha - a^\alpha \psi^\alpha - 2a^\alpha \psi_\alpha W') \\ &\quad - \eta (WW' + \frac{i}{2} \psi^\alpha \psi_\alpha W'') , \quad (33) \end{aligned}$$

where the dot and the prime mean the derivatives with respect to t and q correspondingly. The odd Lagrangian \bar{L} leads to the momenta $p, \pi, \pi^\alpha, p^\alpha$ canonically conjugate in the odd bracket to the coordinates η, q, a^α and ψ^α

$$\{\eta, p\}_1 = \{q, \pi\}_1 = 1;$$

$$\{a^\alpha, \pi^\beta\}_1 = \{\psi^\alpha, p^\beta\}_1 = \delta^{\alpha\beta}.$$

There are the second-class constraints $\varphi^\alpha = \pi^\alpha + \frac{i}{2}\psi^\alpha$, $f^\alpha = -p^\alpha + \frac{i}{2}a^\alpha$

$$\{\varphi^\alpha, f^\beta\}_1 = -i\delta^{\alpha\beta};$$

$$\{\varphi^\alpha, \varphi^\beta\}_1 = \{f^\alpha, f^\beta\}_1 = 0. \quad (34)$$

In terms of the variables $\chi^\alpha = \pi^\alpha - \frac{i}{2}\psi^\alpha$, $g^\alpha = -p^\alpha - \frac{i}{2}a^\alpha$ Dirac's odd bracket from any functions A and B takes the form

$$\{A, B\}_1^{D.B.} = A(\overleftarrow{\partial}_q \overrightarrow{\partial}_\pi - \overleftarrow{\partial}_\pi \overrightarrow{\partial}_q + \overleftarrow{\partial}_\eta \overrightarrow{\partial}_p - \overleftarrow{\partial}_p \overrightarrow{\partial}_\eta + i\overleftarrow{\partial}_{\chi^\alpha} \overrightarrow{\partial}_{g^\alpha} - i\overleftarrow{\partial}_{g^\alpha} \overrightarrow{\partial}_{\chi^\alpha})B, \quad (35)$$

The total odd Hamiltonian following from (33), if subjected to the second-class constraints $\varphi^\alpha = 0$ and $f^\alpha = 0$, takes the form

$$\bar{H} = p\pi + \eta(WW' + \frac{i}{2}\chi_\alpha\chi^\alpha W'') + ig_\alpha\chi^\alpha W' \quad (36)$$

and in Dirac's bracket (35) gives Hamilton's equations $\dot{z}^a = \{z^a, \bar{H}\}_1^{D.B.}$ for the independent phase variables $z^a = (q, p, \chi^\alpha; \eta, \pi, g^\alpha)$

$$\dot{q} = p,$$

$$\dot{p} = -WW' - \frac{i}{2}\chi_\alpha\chi^\alpha W'',$$

$$\dot{\chi}^\alpha = \chi_\alpha W'; \quad (37)$$

$$\dot{\eta} = \pi,$$

$$\dot{\pi} = -\left\{\frac{\eta}{2}[(W^2)'' + \chi_\alpha\chi^\alpha W'''] + ig_\alpha\chi^\alpha W''\right\},$$

$$\dot{g}^\alpha = g_\alpha W' + \eta\chi_\alpha W''. \quad (38)$$

Equations (37) are Hamilton's equations for Witten's supersymmetric mechanics [21] in its classical version [11] which can be derived by means of Dirac's even bracket

$$\{A, B\}_0^{D.B.} = \{A, B\}_0 - i\{A, \varphi^\alpha\}_0\{\varphi^\alpha, B\}_0 = A\left(\overleftarrow{\partial}_q \overrightarrow{\partial}_p - \overleftarrow{\partial}_p \overrightarrow{\partial}_q + i\overleftarrow{\partial}_{\chi^\alpha} \overrightarrow{\partial}_{\chi^\alpha}\right)B \quad (39)$$

with the help of the even Hamiltonian H

$$H = \frac{p^2 + W^2(q)}{2} + \frac{i}{2}\chi_\alpha\chi^\alpha W'(q), \quad (40)$$

which both follow from the $N = 2$ supersymmetric action with the Grassmann-even Lagrangian $L(g(L) = 0)$ (see, for example, [22])

$$S = \frac{1}{4} \int dt d\theta_2 d\theta_1 [D^\alpha \Phi D_\alpha \Phi + 2iV(\Phi)] = \int dt L,$$

where $V'(\Phi) = 2W(\Phi)$ and the even Lagrangian after exclusion of the auxiliary field F is

$$L = \frac{1}{2} [\dot{q}^2 + i(\psi^\alpha \dot{\psi}^\alpha + \psi_\alpha \dot{\psi}^\alpha W') - W^2]. \quad (41)$$

The momenta canonically conjugate in the even bracket to the coordinates q and ψ^α

$$\{q, p\}_0 = 1; \quad \{\psi^\alpha, \pi^\beta\}_0 = -\delta^{\alpha\beta},$$

following from the even Lagrangian (41), lead to the second-class constraints

$$\varphi^\alpha = \pi^\alpha + \frac{i}{2}\psi^\alpha; \quad \{\varphi^\alpha, \varphi^\beta\}_0 = -i\delta^{\alpha\beta}, \quad (42)$$

which commute in the even bracket with the variables $\chi^\alpha = \pi^\alpha - \frac{i}{2}\psi^\alpha$, entering into the definitions for the even Dirac bracket (39) and the even Hamiltonian (40). The even Hamiltonian (40) follows with the use of the second-class constraints restriction $\varphi^\alpha = 0$ from the total even Hamiltonian corresponding to the Lagrangian (41).

Equation (38) can be obtained by taking the exterior differential d of the Hamilton equations (37) for the Witten mechanics and performing the map λ :

$$dq \rightarrow \eta, \quad dp \rightarrow \pi, \quad d\chi^\alpha \rightarrow g^\alpha, \quad d\psi^\alpha \rightarrow a^\alpha,$$

$$d\pi^\alpha \rightarrow -p^\alpha, \quad dF \rightarrow \Xi, \quad d\varphi^\alpha \rightarrow f^\alpha. \quad (43)$$

We identify the grading of d with the Grassmann grading g of the quantities θ^α ($g(d) = g(\theta^\alpha) = 1$), i.e., $g(dx^a) = g(x^a) + 1$. The composition $\lambda \circ d$ of the maps λ and d renders the even Hamiltonian (40) into the odd one (36): $dH \xrightarrow{\lambda} \bar{H}$.

The inter-relation of the brackets (39), (35) and of the corresponding to them Hamiltonians (40), (36) can be described by the following scheme. If we have a Hamilton system described in the bracket $\{A, B\}_\epsilon$ by means of the Hamiltonian \bar{H}

$$\dot{x}^a = \{x^a, \bar{H}\}_\epsilon = \bar{\omega}^{ab} \frac{\partial \bar{H}}{\partial x^b}, \quad (44)$$

where ϵ ($\epsilon = 0, 1$) is the Grassmann parity of both the bracket and the Hamiltonian, then the Hamilton equations for the phase coordinates x^a and the equations for their differentials dx^a , obtaining by a differentiation of equations (44), can be reproduced by the following bracket of the opposite Grassmann parity

$$\begin{aligned} \{A, B\}_{\epsilon+1} = & A \left[\frac{\overleftarrow{\partial}}{\partial x^a} \bar{\omega}^{ab} \frac{\overrightarrow{\partial}}{\partial (dx^b)} \right. \\ & + (-1)^{g(a)+\epsilon} \frac{\overleftarrow{\partial}}{\partial (dx^a)} \bar{\omega}^{ab} \frac{\overrightarrow{\partial}}{\partial x^b} \\ & \left. + \frac{\overleftarrow{\partial}}{\partial (dx^a)} (d \bar{\omega}^{ab}) \frac{\overrightarrow{\partial}}{\partial (dx^b)} \right] B \end{aligned} \quad (45)$$

with the help of the Hamiltonian $d\bar{H}$ ($g(d\bar{H}) = \epsilon + 1$), that is,

$$\dot{x}^a = \{x^a, \bar{H}\}_\epsilon = \{x^a, d\bar{H}\}_{\epsilon+1};$$

$$\dot{dx}^a = d(\{x^a, \bar{H}\}_\epsilon) = \{dx^a, d\bar{H}\}_{\epsilon+1}.$$

In connection with a similar scheme see also the paper [23]. Note also natural appearance of the odd bracket under exterior differentiation of the equations of motion in [13].

There is also an interconnection between Lagrange's equations corresponding to the Lagrangians of the different Grassmann parities L (41) and \bar{L} (33), because the odd Lagrangian \bar{L} (33) is related by means of the redefinition λ (43) with the exterior differential dL of the even Lagrangian (41). Indeed, Lagrange's equations for

the Lagrangian $\bar{L}(q^a, \dot{q}^a)$ with the Grassmann parity ϵ can be written in the two equivalent forms

$$\frac{d}{dt} \left(\frac{\partial \bar{L}}{\partial \dot{q}^a} \right) - \frac{\partial \bar{L}}{\partial q^a} = 0 \quad (46)$$

$$\frac{d}{dt} \left(\frac{\partial (d\bar{L})}{\partial (dq^a)} \right) - \frac{\partial (d\bar{L})}{\partial (dq^a)} = 0, \quad (47)$$

while the equations obtained by taking the differential of (46) have the form

$$\frac{d}{dt} \left(\frac{\partial (d\bar{L})}{\partial \dot{q}^a} \right) - \frac{\partial (d\bar{L})}{\partial \dot{q}^a} = 0. \quad (48)$$

Equations (47), (48) can be considered as Lagrange's equations for the system described in the configuration space q^a, dq^a by the Lagrangian $d\bar{L}$ of the Grassmann parity $\epsilon + 1$. If the Lagrangian \bar{L} has the constraints $\varphi^i(q^a, p_a = \partial \bar{L} / \partial \dot{q}^a)$ satisfying the relations in the bracket corresponding to \bar{L}

$$\{\varphi^i, \varphi^k\}_\epsilon = f^{ik},$$

then the Lagrangian $d\bar{L}$ will possess the constraints $\varphi^i(q^a, p_a = \partial (d\bar{L}) / \partial (dq^a))$, coinciding with those following from \bar{L} , and $d\varphi^i$ obeying the relations

$$\{\varphi^i, d\varphi^k\}_{\epsilon+1} = f^{ik}; \quad \{\varphi^i, \varphi^k\}_{\epsilon+1} = 0,$$

which follow from the related with $d\bar{L}$ bracket expression (45) that in the case is without the last term in their right-hand side (cf., e.g., equations (42) with (34)).

Thus, it is shown that for the given formulation of the dynamics (either in Hamilton's or in Lagrange's approach) with the equations of motion for the dynamical variables x^a we can construct, by using the exterior differential, such a formulation, having the opposite Grassmann parity, that reproduces the former equations for x^a and gives, besides, the equations for their differentials dx^a .

3. QUANTUM ASPECT OF THE ODD BRACKET

3.1. Quantum representations of the odd Poisson bracket

The procedure of the odd-bracket canonical quantization given in [15,16] resides in splitting all the canonical variables into two sets, in the division of all the functions dependent on the canonical variables into classes, and in the introduction of the quantum multiplication $*$, which is either the common product or the bracket composition, in dependence on what the classes the co-factors belong to. Under this, one of the classes has to contain the normalized wave functions, and the result of the multiplication $*$ for any quantity on the wave function Ψ must belong to the class containing Ψ . This procedure is the generalization on the odd bracket case of the canonical quantization rules for the usual Poisson bracket $\{\dots, \dots\}_{\text{Pois}}$, which, for example, in the coordinate representation for the canonical variables q and p are defined as

$$q * \Psi(q) = q\Psi(q) ,$$

$$p * \Psi(q) = i\hbar\{p, \Psi(q)\}_{\text{Pois}} = -i\hbar\frac{\partial\Psi}{\partial q} ,$$

where $\Psi(q)$ is the normalized wave function depending on the coordinate q .

In [15,16] two nonequivalent odd-bracket quantum representations for the canonical variables were obtained by using two different ways of the function division. But these ways do not exhaust all the possibilities. In [17] a more general way of the division is proposed, which contains as the limiting cases the ones given in [15,16].

Let us build quantum representations for an arbitrary graded bracket under its canonical quantization. To this end, all canonical variables are split into two equal in the number sets, so that none of them should contain the pairs of canonical conjugates. Note that to make such a splitting possible for the even bracket (1), the transition has to be done from the real canonical self-conjugate odd variables to some pairs of odd variables, which simultaneously are complex and canonical conjugate to each other. Composing

from the integer degrees of the variables from the one set (we call it the first set) the monomials of the odd $2s+1$ and even $2s$ uniformity degrees and multiplying them by the arbitrary functions dependent on the variables from the other (second) set, we thus divide all the functions of the canonical variables into the classes designated as $\overset{\epsilon}{O}_s$ and $\overset{\epsilon}{E}_s$, respectively. For instance, in the general case the odd-bracket canonical variables can be split, so that the first set would contain the even y_i ($i = 1, \dots, n \leq N$) and odd $\eta^{n+\alpha}$ ($\alpha = 1, \dots, N-n$) variables, while the second set would involve the rest variables [17]. Then the classes of the functions obtained under this splitting have the form

$$\overset{1}{O}_s = (y_i, \eta^{n+\alpha})^{2s+1} f(\eta^i, y_{n+\alpha}) ;$$

$$\overset{1}{E}_s = (y_i, \eta^{n+\alpha})^{2s} f(\eta^i, y_{n+\alpha}) ,$$

where the factors before the arbitrary function $f(\eta^i, y_{n+\alpha})$ denote the monomials having the uniformity degrees indicated in the exponents. These classes satisfy the corresponding bracket relations

$$\begin{aligned} \{\overset{\epsilon}{O}_s, \overset{\epsilon}{O}_{s'}\}_{\epsilon} &= \overset{\epsilon}{O}_{s+s'} ; & \{\overset{\epsilon}{O}_s, \overset{\epsilon}{E}_{s'}\}_{\epsilon} &= \overset{\epsilon}{E}_{s+s'} ; \\ \{\overset{\epsilon}{E}_s, \overset{\epsilon}{E}_{s'}\}_{\epsilon} &= \overset{\epsilon}{O}_{s+s'-1} , \end{aligned} \quad (49)$$

and the relations of the ordinary Grassmann multiplication

$$\begin{aligned} \overset{\epsilon}{O}_s \cdot \overset{\epsilon}{O}_{s'} &= \overset{\epsilon}{O}_{s+s'+1} ; & \overset{\epsilon}{O}_s \cdot \overset{\epsilon}{E}_{s'} &= \overset{\epsilon}{O}_{s+s'} ; \\ \overset{\epsilon}{E}_s \cdot \overset{\epsilon}{E}_{s'} &= \overset{\epsilon}{E}_{s+s'} . \end{aligned} \quad (50)$$

It follows from (49), (50), that $\overset{\epsilon}{O} = \{\overset{\epsilon}{O}_s\}$ and $\overset{\epsilon}{E} = \{\overset{\epsilon}{E}_s\}$ form a superalgebra with respect to the addition and the quantum multiplication $\overset{\epsilon}{*}$ ($\epsilon = 0, 1$) defined for the corresponding bracket as

$$\begin{aligned} \overset{\epsilon'}{O} \overset{\epsilon''}{*} \overset{\epsilon'''}{O} &= \{\overset{\epsilon'}{O}, \overset{\epsilon'''}{O}\}_{\epsilon} \in \overset{\epsilon}{O} ; \\ \overset{\epsilon'}{O} \overset{\epsilon''}{*} \overset{\epsilon'''}{E} &= \{\overset{\epsilon'}{O}, \overset{\epsilon'''}{E}\}_{\epsilon} \in \overset{\epsilon}{E} ; \\ \overset{\epsilon'}{E} \overset{\epsilon''}{*} \overset{\epsilon'''}{E} &= \overset{\epsilon'}{E} \cdot \overset{\epsilon'''}{E} \in \overset{\epsilon}{E} , \end{aligned} \quad (51)$$

where $\overset{\epsilon'}{O}, \overset{\epsilon''}{O} \in \overset{\epsilon}{O}$ and $\overset{\epsilon'}{E}, \overset{\epsilon''}{E} \in \overset{\epsilon}{E}$. Note, that the classes $\overset{\epsilon}{O}_0$ and $\overset{\epsilon}{E}_0$ form the sub-superalgebra. In terms of the quantum grading $q_\epsilon(A)$ of any quantity A

$$q_\epsilon(A) = \begin{cases} g_\epsilon(A), & \text{for } A \in \overset{\epsilon}{O}; \\ g(A), & \text{for } A \in \overset{\epsilon}{E}, \end{cases}$$

introduced for the appropriate bracket, the grading and symmetry properties of the quantum multiplication $\overset{\epsilon}{*}$, arising from the corresponding properties for the bracket (3), (4) and Grassmann composition of any two quantities A and B , are uniformly written as

$$q_\epsilon(A \overset{\epsilon}{*} B) = q_\epsilon(A) + q_\epsilon(B), \quad (52)$$

$$\overset{\epsilon'}{O} \overset{\epsilon}{*} \overset{\epsilon''}{O} = -(-1)^{q_\epsilon(\overset{\epsilon'}{O})q_\epsilon(\overset{\epsilon''}{O})} \overset{\epsilon''}{O} \overset{\epsilon}{*} \overset{\epsilon'}{O}, \quad (53)$$

$$\overset{\epsilon'}{E} \overset{\epsilon}{*} \overset{\epsilon''}{E} = (-1)^{q_\epsilon(\overset{\epsilon'}{E})q_\epsilon(\overset{\epsilon''}{E})} \overset{\epsilon''}{E} \overset{\epsilon}{*} \overset{\epsilon'}{E}. \quad (54)$$

With the use of the quantum multiplication $\overset{\epsilon}{*}$ and the quantum grading q_ϵ , let us define for any two quantities A, B the quantum bracket ((anti)commutator) $[A, B]_\epsilon$ (under its action on the wave function Ψ that is considered to belong to the class E) in the form [15–17]

$$[A, B]_\epsilon \overset{\epsilon}{*} \Psi = A \overset{\epsilon}{*} (B \overset{\epsilon}{*} \Psi) - (-1)^{q_\epsilon(A)q_\epsilon(B)} B \overset{\epsilon}{*} (A \overset{\epsilon}{*} \Psi). \quad (55)$$

If $A, B \in \overset{\epsilon}{E}$, then, due to (54), the quantum bracket between them equals zero. In particular, the wave functions are (anti)commutative. If A or both of the quantities A and B belong to the class $\overset{\epsilon}{O}$, then in the first case, due to the Leibnitz rule (6), and in the second one, because of the Jacobi identities (5), the relation follows from the definitions (51) and (55)

$$[A, B]_\epsilon \overset{\epsilon}{*} \Psi = \{A, B\}_\epsilon \overset{\epsilon}{*} \Psi = (A \overset{\epsilon}{*} B) \overset{\epsilon}{*} \Psi,$$

that establishes the connection between the classical and quantum brackets of the corresponding Grassmann parity. Note, that the quantization procedure also admits the reduction to $O_o \cup E_o$.

The grading q_ϵ determines the symmetry properties of the quantum bracket (55). Under above-mentioned splitting of the odd-bracket canonical

variables into two sets, the grading q_1 equals unity for the variables $y_i \in \overset{1}{O}$, $\eta^i \in \overset{1}{E}$ ($i = 1, \dots, n \leq N$) and equal to zero for the rest canonical variables $y_{n+\alpha} \in \overset{1}{E}$, $\eta^{n+\alpha} \in \overset{1}{O}$ ($\alpha = 1, \dots, N - n$). Therefore, in this case the quantum odd bracket is represented with the anticommutators between the quantities y_i, η^i and with the commutators for the remaining relations of the canonical variables. If the roles of the first and the second sets of the canonical variables change, then the quantum bracket is represented with the anticommutators between $y_{n+\alpha}, \eta^{n+\alpha}$ and with the commutators in the other relations. In [15,16] the odd-bracket quantum representations were obtained for the cases $n = 0, N$, containing, respectively, only commutators or anticommutators.

3.2. Quantization of the systems with the odd Poisson bracket

As the simplest example of using of the odd-bracket quantum representations under the quantization of the classical systems based on the odd bracket [17], let us consider the one-dimensional supersymmetric oscillator, whose phase superspace x^A contains a pair of even q, p and a pair of odd η^1, η^2 real canonical coordinates. In terms of more suitable complex coordinates $z = (p - iq)/\sqrt{2}$, $\eta = (\eta^1 - i\eta^2)/\sqrt{2}$ and their complex conjugates $\bar{z}, \bar{\eta}$, the even bracket is written as

$$\{A, B\}_0 = iA[\overset{\leftarrow}{\partial}_{\bar{z}}\overset{\rightarrow}{\partial}_z - \overset{\leftarrow}{\partial}_z\overset{\rightarrow}{\partial}_{\bar{z}} - (\overset{\leftarrow}{\partial}_{\bar{\eta}}\overset{\rightarrow}{\partial}_\eta + \overset{\leftarrow}{\partial}_\eta\overset{\rightarrow}{\partial}_{\bar{\eta}})]B \quad (56)$$

and the even Hamiltonian H , the supercharges Q_1, Q_2 and the fermionic charge F have the forms

$$H = z\bar{z} + \bar{\eta}\eta; \quad Q_1 = \bar{z}\eta + z\bar{\eta}; \quad Q_2 = i(\bar{z}\eta - z\bar{\eta}); \quad F = \eta\bar{\eta}. \quad (57)$$

The odd Hamiltonian \bar{H} and the appropriate odd bracket, which reproduce the same Hamilton equations of motion, as those resulting from (56) with the even Hamiltonian H (57), i.e., which satisfy the condition (12), can be taken as $\bar{H} = Q_1$ and

$$\{A, B\}_1 = iA(\overset{\leftarrow}{\partial}_{\bar{z}}\overset{\rightarrow}{\partial}_\eta - \overset{\leftarrow}{\partial}_\eta\overset{\rightarrow}{\partial}_{\bar{z}} + \overset{\leftarrow}{\partial}_{\bar{\eta}}\overset{\rightarrow}{\partial}_z - \overset{\leftarrow}{\partial}_z\overset{\rightarrow}{\partial}_{\bar{\eta}})B. \quad (58)$$

The complex variables have the advantage over the real ones, because with their use the splitting of the canonical variables into two sets $\bar{z}, \bar{\eta}$ and z, η satisfies simultaneously the requirements necessary for the quantization both of the brackets (56), (58). Besides, any of the vector fields $\overset{\epsilon}{X}_{A_i} = -i\{A_i, \dots\}_\epsilon$ for the quantities $\{A_i\} = (H, Q_1, Q_2, F)$, describing the dynamics and the symmetry of the system under consideration, is split into the sum of two differential operators dependent on either $\bar{z}, \bar{\eta}$ or z, η . For instance, from (56)-(58) we have

$$\overset{0}{X}_H = \overset{1}{X}_H = z\partial_z + \eta\partial_\eta - \bar{z}\partial_{\bar{z}} - \bar{\eta}\partial_{\bar{\eta}}. \quad (59)$$

The diagonalization does not take place in terms of the variables $x^A = (q, p; \eta^1, \eta^2)$.

In accordance with the above-mentioned splitting of the complex variables, we can perform one of the two possible divisions all of the functions into the classes, which are common for both of the brackets (56), (58), playing a crucial role under their canonical quantization and leading to the same quantum dynamics for the system under consideration. If $\bar{z}, \bar{\eta}$ are attributed to the first set, then the corresponding function division is

$$\overset{\epsilon}{E}_s = (\bar{z}\bar{\eta})^{2s+1} f(z, \eta); \quad \overset{\epsilon}{E}_s = (\bar{z}\bar{\eta})^{2s} f(z, \eta).$$

If we restrict ourselves to the classes O_o and E_o , then $\Psi \in E_o$ and depends only on z, η and $A_i \in O_o$. According to the definition (51), the results of the quantum multiplications $\overset{*}{_1}$ and $\overset{*}{_0}$ of $z, \eta \in E_o$ and $\bar{z}, \bar{\eta} \in O_o$ on the wave function Ψ are

$$z \overset{*}{_1} \Psi = z \overset{*}{_0} \Psi = z \cdot \Psi;$$

$$\bar{\eta} \overset{*}{_1} \Psi = \bar{z} \overset{*}{_0} \Psi = \partial_z \Psi;$$

$$\eta \overset{*}{_1} \Psi = \eta \overset{*}{_0} \Psi = \eta \cdot \Psi;$$

$$\bar{z} \overset{*}{_1} \Psi = -\bar{\eta} \overset{*}{_0} \Psi = \partial_\eta \Psi. \quad (60)$$

The positive definite scalar product of the wave functions $\Psi_1(z, \eta)$ and $\Psi_2(z, \eta)$ can be determined in the form [24]

$$(\Psi_1, \Psi_2) = \frac{1}{\pi} \int \exp[-(|z|^2 + \bar{\theta}\eta)] \Psi_1(z, \eta) \times [\Psi_2(z, \theta)]^+ d\bar{\theta} d\eta d(Re z) d(Im z), \quad (61)$$

where θ is the auxiliary complex Grassmann quantity anticommuting with η , and the integration over the real and imaginary components of z is performed in the limits $(-\infty, \infty)$. It is easy to see that with respect to the scalar product (61) the pairs of the canonical variables, being Hermitian conjugated to each other under the multiplication $\overset{*}{_1}$, are $z, \bar{\eta}$ and \bar{z}, η , but under $\overset{*}{_0}$ are z, \bar{z} and $\eta, -\bar{\eta}$.

In order to have the action of the Hamiltonian operator, obtained from the system quantization, on the wave function, we need, as it is well known, to replace the canonical variables in the classical Hamiltonian by the respective operators or, which is the same, to define their action with the help of the corresponding quantum multiplication $\overset{*}{_1}$. In this connection, in view of (59), (60), we see that the self-consistent quantum Hamilton operators in the even and odd cases, being in agreement with the classical expressions (57) for the equivalent Hamiltonians H and \bar{H} and giving the same result at the action on $\Psi(z, \eta)$, will be respectively

$$H \overset{*}{_0} \Psi = z \overset{*}{_0} (\bar{z} \overset{*}{_0} \Psi) - \eta \overset{*}{_0} (\bar{\eta} \overset{*}{_0} \Psi); \quad (62)$$

$$\bar{H} \overset{*}{_1} \Psi = z \overset{*}{_1} (\bar{\eta} \overset{*}{_1} \Psi) + \eta \overset{*}{_1} (\bar{z} \overset{*}{_1} \Psi). \quad (63)$$

The Hamiltonians (62), (63) are Hermitian relative to the scalar product (61) and both, due to (60), are reduced to the Hamilton operator for the one-dimensional supersymmetric oscillator $H = a^+ a + b^+ b$ expressed in terms of the creation and annihilation operators for the bosons $a^+ = z$, $a = \partial_z$ and fermions $b^+ = \eta$, $b = \partial_\eta$ respectively, in the Fock-Bargmann representation (see, for example [25]). The normalized with respect to (61) eigenfunctions $\Psi_{k,n}(z, \eta)$ of the Hamiltonians (62), (63), corresponding to energy eigenvalues $E_{k,n} = k + n$ ($k = 0, 1; n = 0, 1, \dots, \infty$) have the form

$$\Psi_{k,n}(z, \eta) = \frac{1}{\sqrt{n!}} (\eta \overset{*}{_1})^k (z \overset{*}{_1})^n 1.$$

Note, that another equivalent representation of the quantum supersymmetric oscillator can be obtained, if the canonical variables z, η are chosen as the first set. Let us also note that the consideration described above can be extended to the

quantization of a set non-interacting supersymmetric oscillators by supplement all the canonical variables $z, \bar{z}, \eta, \bar{\eta}$ with the index i ($i = 1, \dots, N$) over which a summation have to be performed in the bilinear combinations of the variables in all the formulas of this subsection.

Thus, we have demonstrated that the use of the quantum representations found for the odd bracket [17] leads to the self-consistent quantization of the classical Hamilton systems based on this bracket. We should apparently expect that these representations are also applicable for the quantization of more complicated classical systems with the odd bracket.

3.3. Composite spinor structure of space-time

The odd-bracket quantum representation with $n = N$ for the canonical variables $y_{\alpha i}, \eta^{\beta k}$

$$\{y_{\alpha i}, \eta^{\beta k}\}_1 = \delta_{\alpha}^{\beta} \delta_i^k, \quad \{y_{\alpha i}, y_{\beta k}\}_1 = \{\eta^{\alpha i}, \eta^{\beta k}\}_1 = 0 \quad (64)$$

with the spinor indices α, β of the space-time symmetry and with the indices i, k of the internal symmetry can be applied for the realization of the idea of a composite spinor structure of space-time [15]. Here we illustrate the realization of this idea on the example of superspace of the orthosymplectic groups $OSp(N, 2k)$ with $k = 1, 2$, which are $SO(N)$ -extended supergeneralizations of the de Sitter groups $O(k, k+1)$, which admit real spinor representations. For the quantum representation with $n = N$ the division of functions of the canonical variables has the form

$$\overset{1}{O}_s = y^{2s+1} f(\eta); \quad \overset{1}{E}_s = y^{2s} f(\eta).$$

The superspace coordinates $z^a = (x^\mu, \theta^{\alpha i})$ and the wave function $\Psi(z)$ will be constructed from quantities belonging to $\overset{1}{E}_0$, while the dynamical variables describing the symmetry of the coordinates z^a will be constructed from the quantities of the class $\overset{1}{O}_0$.

Using the relations (64) and (6), we can convince ourselves that the generators of supertranslations (here we follow the notations of [15])

$$Q_{\alpha i} = y_{\alpha i} - i\kappa(\bar{\eta}_i y_k) \eta_{\alpha}^k \quad (65)$$

together with the generators

$$L_{\alpha\beta} = -2iy_{i(\alpha} \eta_{\beta)}^i \quad (66)$$

of the group $Sp(2k, R)$ which henceforth plays the role of the group of the space-time symmetry, and the generators

$$I_{ik} = -3i(\bar{y}_{[i} \eta_{k]}) \quad (67)$$

of the internal-symmetry group $SO(N)$ form, with respect to the odd bracket (64), the superalgebra of the group $OSp(N, 2k)$. In the definition (65) for the supertranslation generators we introduced a dimensional parameter κ with the meaning of the inverse radius of curvature of the de Sitter space. Letting κ tend to zero, we go over to superalgebras of N -extended Poincaré supergroups: a two-dimensional superalgebra when $k = 1$, and a four-dimensional superalgebra when $k = 2$.

We now show how, with the aid of the representations (65)–(67) for the generators of the group $OSp(N, 2k)$ ($k = 1, 2$), it is possible to construct in terms of the quantities $y_{\alpha i}$ and $\eta^{\beta k}$ the usual coordinates x^μ and $\theta^{\alpha i}$ of de Sitter superspaces. It follows from the relations (64), (6) and (65) that the spinors $\eta^{\alpha i}$ and $y_{\alpha i}$ are transformed by the generators of supertranslations as:

$$\delta \eta^{\alpha i} = \{(\bar{a}Q), \eta^{\alpha i}\}_1 = a^{\alpha i} + i\kappa(\bar{a}^k \eta_i) \eta_{\alpha}^k, \quad (68)$$

$$\delta y_{\alpha i} = \{(\bar{a}Q), y_{\alpha i}\}_1 = -i\kappa[a_{\alpha}^k (\bar{\eta}_k y_i) + (\bar{a}_i \eta_k) y_{\alpha}^k], \quad (69)$$

where $a^{\alpha i}$ are anticommuting parameters. The relation (68) is the transformation law under supertranslations for the coordinates $\eta^{\alpha i}$ of the homogeneous space $OSp(N, 2k)/SO(N) \otimes Sp(2k, R)$, and these coordinates can be identified with the conventionally used Grassmann variables $\theta^{\alpha i}$:

$$\theta^{\alpha i} = \eta^{\alpha i}, \quad (70)$$

whereas (69) can be represented in the form of a transformation of a 1-form field specified on this homogeneous space:

$$y'_{\alpha i} = \frac{\partial \eta^{\beta k}}{\partial \eta'^{\alpha i}} y_{\beta k}.$$

We shall transit from the variables $y_{\alpha i}$ to variables $\tilde{y}_{\alpha i}(y, \eta)$ that are linear in $y_{\alpha i}$ and have the following transformation law with respect to generators $Q_{\alpha i}$

$$\delta \tilde{y}_{\alpha i} = \{(\bar{a}Q), \tilde{y}_{\alpha i}\}_1 = i(\lambda_{\alpha\beta} \tilde{y}_i^\beta + \lambda_{ik} \tilde{y}_\alpha^k), \quad (71)$$

where $\lambda_{\alpha\beta}$ is a symmetric matrix and λ_{ik} is an antisymmetric one, both of which depend on the parameters $a^{\alpha i}$ and the coordinates $\eta^{\alpha i}$. The form of the functions $\tilde{y}_{\alpha i}(y, \eta)$ and the matrices $\lambda_{\alpha\beta}(a, \eta)$ and $\lambda_{ik}(a, \eta)$ depends on the values of N and k . For the following analysis we shall need only the fact of the existence of such variables $\tilde{y}_{\alpha i}$ and the structure of their transformation law (71), which determines in infinitesimal form that representation of the group $OSp(N, 2k)$ which is induced by the representation of its even subgroup $SO(N) \otimes Sp(2k, R)$ that is the vector representation with respect to $SO(N)$ and the defining representation with respect to the groups $Sp(2k, R)$.

For the construction of the space-time coordinates x^μ from the quantities $\tilde{y}_{\alpha i}$ we define supertransformations of x^μ using that representation of the group $OSp(N, 2k)$ which is induced by the vector representation of the de Sitter group $SO(k, k+1)$. With respect to the internal-symmetry group $SO(N)$ the inducing representation is taken to be a scalar. By taking into account the transformation law (71) for the quantities $\tilde{y}_{\alpha i}$, we shall construct from them a real vector V^A with respect to the group $SO(k, k+1)$. The simplest expression for V^A , containing the lowest powers of $\tilde{y}_{\alpha i}$, has the form

$$V^A = (\tilde{y}_i \Gamma^A \tilde{y}_k) g^{ik} \quad (72)$$

for the case $k = 1$, and

$$V^A = (\tilde{y}_i \Gamma^A \tilde{y}_k) ((\tilde{y})^i \tilde{y}^k) \quad (73)$$

for the case $k = 2$.

One should note the remarkable fact that in the case $k = 2$, because of the antisymmetry of the matrices $C^{-1}\Gamma$, the vector V^A can be constructed only if we introduce an internal-symmetry group $SO(N)$ with $N \geq 2$, whereas for $k = 1$ these matrices are symmetric and V^A can be constructed in the case $N = 1$. However, in this case $V^A V_A = 0$, i.e., such a vector is isotropic.

We shall define the coordinates x^μ for both values of k in terms of the corresponding expressions for V^A in the form

$$x^\mu = \frac{V^\mu}{\kappa V^5} \quad (74)$$

with the condition that the quantity $V^2 = V^A V_A$, which is invariant of $OSp(N, 2k)$, has the sign which, for each value of k , leads to the correct signature of the space-time metric, i.e., V^2 should be positive when $k = 1$ and negative when $k = 2$. In the cases that we have considered we did not succeed in satisfying these conditions for compact internal-symmetry groups $SO(N)$. By going over to the non-compact groups $SO(n, m)$ with $n + m = N$ it is possible, albeit not for all values of N , to achieve the correct signature of the space-time metric. It is possible that this circumstance is not accidental and is connected with the well-known fact that in $N \geq 4$ extended supergravities the scalar fields form homogeneous non-compact spaces [26].

We note one feature that appears to us to be curious. If besides the coordinates x^μ and $\theta^{\alpha i}$ we construct composite coordinates u^{ik} , corresponding to the internal-symmetry generators I_{ik} , then the quadratic equation

$$2k + \frac{N(N-1)}{2} = 2^k N,$$

reflecting the equality of the numbers of degrees of freedom of the coordinates x^μ and u^{ik} and of their constituent spinor $y_{\alpha i}$, has the solutions $N = 1, 4$ for $k = 1$ and $N = 1, 8$ for $k = 2$. We must discard the solutions with $N = 1$, since for these it is impossible to construct x^μ with the appropriate signature and non-singular character. The remaining solutions imply that for the composite spinor structure of the superspace the value of N is subject to a restriction that coincides with the restriction that follows from consideration of the maximal multiplet of extended supergravity.

Using the transformation law (71) for $\tilde{y}_{\alpha i}$ and the expressions (72)–(74) for the coordinates x^μ , we can obtain the transformation of x^μ under supertranslations; the form of this transformation is determined by the structure of the matrices $\lambda_{\alpha\beta}$, i.e., depends on the values of N and k . As an example we shall give the transformation of x^μ for

the case of the group $OSp(1, 1; 2)$:

$$\begin{aligned} \delta x^\mu &= \{(\bar{a}Q), x^\mu\}_1 = -i(\bar{a}^k \gamma^\mu \eta_k) \\ &+ 2i\kappa(\bar{a}^k \sigma^{\mu\nu} \eta_k) x_\nu + i\kappa^2(\bar{a}^k \gamma^\nu \eta_k) x_\nu x^\mu, \end{aligned} \quad (75)$$

where the term independent of κ for the other values of N and k will be the same as in (75), whereas the terms containing κ can be different.

From the relations (72)–(74) we obtain the following bracket relations for angular momentum

$$M_{\mu\nu} = -i(\bar{y}^k \sigma_{\mu\nu} \eta_k),$$

the momentum

$$P_\mu = \frac{i}{2} \kappa(\bar{y}^k \gamma_\mu \eta_k)$$

and the generators I_{ik} with coordinates x^μ and $\theta^{\alpha i}$:

$$\{P_\mu, x^\nu\}_1 = -i(\delta_\mu^\nu - \kappa^2 x_\mu x^\nu),$$

$$\{M_{\mu\nu}, x^\rho\}_1 = -i(x_\mu \delta_\nu^\rho - x_\nu \delta_\mu^\rho),$$

$$\{I_{ik}, x^\mu\}_1 = 0,$$

$$\{P_\mu, \theta^{\alpha i}\}_1 = \frac{i}{2} \kappa(\bar{\theta}^i \gamma_\mu)^\alpha,$$

$$\{M_{\mu\nu}, \theta^{\alpha i}\}_1 = -i(\bar{\theta}^i \sigma_{\mu\nu})^\alpha,$$

$$\{I_{ik}, \theta^{\alpha l}\}_1 = i(\delta_i^l \theta_k^\alpha - \delta_k^l \theta_i^\alpha), \quad (76)$$

where for each value of k it is necessary to take the matrices γ_μ and $\sigma_{\mu\nu}$ corresponding to it.

As κ tends to zero the supergroups $OSp(n, m; 2k)$ go over into extended Poincaré supergroups, and at the same time the relations (68), (75) and (76), with allowance for the relationship between the classical and quantum odd brackets, reproduce, on the composite coordinates $z^a = (x^\mu, \theta^{\alpha i})$ and dynamical quantities $Q_{\alpha i}$, P_μ , $M_{\mu\nu}$ and I_{ik} , describing their symmetry, the algebra of the conventional canonical quantization of the coordinates and dynamical quantities.

4. ALGEBRAIC ASPECT OF THE ODD BRACKET

4.1. Linear odd bracket on Grassmann algebra

In this section we show that the linear odd bracket can be realized solely on the Grassmann variables [19]. It will be also shown that with such a bracket, that corresponds to a semi-simple Lie group, both a Grassmann-odd Casimir function and invariant (with respect to this group) nilpotent differential operators of the first, second and third orders are naturally related and enter into a finite-dimensional Lie superalgebra. It will be also pointed out that some of the quantities, forming this Lie superalgebra, can be treated as the BRST charge, Δ -operator and operator for the ghost number.

There is a well-known linear even Poisson bracket given in terms of the commuting variables X_α

$$\{X_\alpha, X_\beta\}_0 = c_{\alpha\beta}^\gamma X_\gamma, \quad (\alpha, \beta, \gamma = 1, \dots, N), \quad (77)$$

where constants $c_{\alpha\beta}^\gamma$, because of the main properties of the even Poisson bracket (3)–(6), are antisymmetric in the two lower indices

$$c_{\alpha\beta}^\gamma = -c_{\beta\alpha}^\gamma \quad (78)$$

and obey the conditions

$$c_{\alpha\lambda}^\delta c_{\beta\gamma}^\lambda + c_{\beta\lambda}^\delta c_{\gamma\alpha}^\lambda + c_{\gamma\lambda}^\delta c_{\alpha\beta}^\lambda = 0. \quad (79)$$

The linear even bracket (77) plays a very important role in the theory of Lie groups, Lie algebras, their representations and applications (see, for example, [20, 27]). The bracket (77) can be realized in a canonical even Poisson bracket

$$\{A, B\}_0 = A \sum_{\alpha=1}^N \left(\overleftarrow{\partial}_{q^\alpha} \overrightarrow{\partial}_{p_\alpha} - \overleftarrow{\partial}_{p_\alpha} \overrightarrow{\partial}_{q^\alpha} \right) B$$

on the following bilinear functions of coordinates q^α and momenta p_α

$$X_\alpha = c_{\alpha\beta}^\gamma q^\beta p_\gamma$$

if $c_{\alpha\beta}^\gamma$ satisfy the conditions (78), (79) for the structure constants of a Lie group.

As in the Lie algebra case, we can define a symmetric Cartan-Killing tensor

$$g_{\alpha\beta} = g_{\beta\alpha} = c_{\alpha\gamma}^\lambda c_{\beta\lambda}^\gamma \quad (80)$$

and verify with the use of relations (79) an anti-symmetry property of a tensor

$$c_{\alpha\beta\gamma} = c_{\alpha\beta}{}^{\delta} g_{\delta\gamma} = -c_{\alpha\gamma\beta} . \quad (81)$$

By assuming that the Cartan-Killing metric tensor is non-degenerate $\det(g_{\alpha\beta}) \neq 0$ (this case corresponds to the semi-simple Lie group), we can define an inverse tensor $g^{\alpha\beta}$

$$g^{\alpha\beta} g_{\beta\gamma} = \delta_{\gamma}^{\alpha} , \quad (82)$$

with the help of which we are able to build a quantity

$$C = X_{\alpha} X_{\beta} g^{\alpha\beta} ,$$

that, in consequence of relation (81), is for the bracket (77) a Casimir function which annihilates the bracket (77) and is an invariant of the Lie group with the structure constants $c_{\alpha\beta}{}^{\gamma}$ and the generators T_{α}

$$\{X_{\alpha}, C\}_0 = c_{\alpha\beta}{}^{\gamma} X_{\gamma} \partial_{X_{\beta}} C = T_{\alpha} C = 0 .$$

Now let us replace in expression (77) the commuting variables X_{α} by Grassmann variables Θ_{α} ($g(\Theta_{\alpha}) = 1$). Then we obtain a binary composition

$$\{\Theta_{\alpha}, \Theta_{\beta}\}_1 = c_{\alpha\beta}{}^{\gamma} \Theta_{\gamma} , \quad (83)$$

which, due to relations (78) and (79), meets all the properties (3)–(6) of the odd Poisson brackets. It is surprising enough that the odd bracket can be defined solely in terms of the Grassmann variables as well as an even Martin bracket [5]. On the following bilinear functions of canonical variables commuting q^{α} and Grassmann θ_{α}

$$\Theta_{\alpha} = c_{\alpha\beta}{}^{\gamma} q^{\beta} \theta_{\gamma}$$

a canonical odd Poisson bracket

$$\{A, B\}_1 = A \sum_{\alpha=1}^N \left(\overleftarrow{\partial}_{q^{\alpha}} \overrightarrow{\partial}_{\theta_{\alpha}} - \overleftarrow{\partial}_{\theta_{\alpha}} \overrightarrow{\partial}_{q^{\alpha}} \right) B$$

is reduced to the bracket (83) providing that $c_{\alpha\beta}{}^{\gamma}$ obey the conditions (78), (79).

By the way, let us note that we can also construct only on the Grassmann variables a non-linear odd Poisson bracket of the form

$$\{\Theta_{\alpha}, \Theta_{\beta}\}_1 = c_{\alpha\beta}{}^{\gamma\delta\lambda} \Theta_{\gamma} \Theta_{\delta} \Theta_{\lambda} .$$

In order to satisfy the property (4), the constants $c_{\alpha\beta}{}^{\gamma\delta\lambda}$ have to be antisymmetric in the below indices

$$c_{\alpha\beta}{}^{\gamma\delta\lambda} = -c_{\beta\alpha}{}^{\gamma\delta\lambda}$$

and, for the validity of the Jacobi identities (5), they must obey the following conditions

$$\sum_{(\alpha\beta\gamma)} c_{\alpha\beta}{}^{\lambda[\delta_1\delta_2} c_{\lambda\gamma}{}^{\delta_3\delta_4\delta_5]} = 0 ,$$

where square brackets [...] mean an antisymmetrization of the indices in them.

Returning to the linear odd bracket (83) notice, that on functions A, B of Grassmann variables Θ_{α} this bracket has the form

$$\{A, B\}_1 = A \overleftarrow{\partial}_{\Theta_{\alpha}} c_{\alpha\beta}{}^{\gamma} \Theta_{\gamma} \overrightarrow{\partial}_{\Theta_{\beta}} B .$$

The bracket (83) can be either degenerate or non-degenerate in the dependence on whether the matrix $c_{\alpha\beta}{}^{\gamma} \Theta_{\gamma}$ in the indices α, β is degenerate or not. Raising and lowering of the indices α, β , the non-degenerate metric tensors (80), (82) relate with each other the adjoint and co-adjoint representations which are equivalent for a semi-simple Lie group

$$\Theta^{\alpha} = g^{\alpha\beta} \Theta_{\beta} , \quad \partial_{\Theta^{\alpha}} = g_{\alpha\beta} \partial_{\Theta_{\beta}} .$$

Hereafter only the non-degenerate metric tensors (82) will be considered.

By contracting the indices in a product of the Grassmann variables with the upper indices and of the successive Grassmann derivatives, respectively, with the lower indices in (79), we obtain the relations

$$\Theta^{\alpha} \Theta^{\beta} (c_{\alpha\beta}{}^{\lambda} c_{\lambda\gamma}{}^{\delta} + 2c_{\gamma\alpha}{}^{\lambda} c_{\lambda\beta}{}^{\delta}) = 0 , \quad (84)$$

$$\Theta^{\alpha} \Theta^{\beta} \Theta^{\gamma} c_{\alpha\beta}{}^{\lambda} c_{\lambda\gamma}{}^{\delta} = 0 , \quad (85)$$

$$(c_{\alpha\beta}{}^{\lambda} c_{\lambda\gamma}{}^{\delta} + 2c_{\gamma\alpha}{}^{\lambda} c_{\lambda\beta}{}^{\delta}) \partial_{\Theta_{\alpha}} \partial_{\Theta_{\beta}} = 0 , \quad (86)$$

$$c_{\alpha\beta}{}^{\lambda} c_{\lambda\gamma}{}^{\delta} \partial_{\Theta_{\alpha}} \partial_{\Theta_{\beta}} \partial_{\Theta_{\gamma}} = 0 , \quad (87)$$

which will be used later on many times. In particular, taking into account relation (85), we can verify that the linear odd bracket (83) has the following Grassmann-odd nilpotent Casimir function

$$\Delta_{+3} = \frac{1}{\sqrt{3!}} \Theta^{\alpha} \Theta^{\beta} \Theta^{\gamma} c_{\alpha\beta\gamma} , \quad (\Delta_{+3})^2 = 0 , \quad (88)$$

which is an invariant of the Lie group

$$\{\Theta_\alpha, \Delta_{+3}\}_1 = \Theta_\gamma c_{\alpha\beta}^\gamma \partial_{\Theta_\beta} \Delta_{+3} = S_\alpha \Delta_{+3} = 0 \quad (89)$$

with the generators S_α obeying the Lie algebra permutation relations (below $[A, B] = AB - BA$ and $\{A, B\} = AB + BA$)

$$[S_\alpha, S_\beta] = c_{\alpha\beta}^\gamma S_\gamma. \quad (90)$$

It is a well-known fact that, in contrast with the even Poisson bracket, the non-degenerate odd Poisson bracket has one Grassmann-odd nilpotent differential Δ -operator of the second order, in terms of which the main equation has been formulated in the Batalin-Vilkovisky scheme [7] for the quantization of gauge theories in the Lagrangian approach. In a formulation of Hamiltonian dynamics by means of the odd Poisson bracket with the help of a Grassmann-odd Hamiltonian \bar{H} ($g(\bar{H}) = 1$) [8,11,12] this Δ -operator plays also a very important role being used to distinguish the Hamiltonian dynamical systems, for which the Liouville theorem is valid $\Delta\bar{H} = 0$, from those ones, for which this theorem takes no place $\Delta\bar{H} \neq 0$.

Now let us try to build the Δ -operator for the linear odd bracket (83). It is remarkable that, in contrast with the canonical odd Poisson bracket having the only Δ -operator of the second order, we are able to construct at once three Δ -like Grassmann-odd nilpotent operators which are differential operators of the first, second and third orders respectively

$$\Delta_{+1} = \frac{1}{\sqrt{2}} \Theta^\alpha \Theta^\beta c_{\alpha\beta}^\gamma \partial_{\Theta_\gamma}, \quad (\Delta_{+1})^2 = 0; \quad (91)$$

$$\Delta_{-1} = \frac{1}{\sqrt{2}} \Theta_\gamma c_{\alpha\beta}^\gamma \partial_{\Theta_\alpha} \partial_{\Theta_\beta}, \quad (\Delta_{-1})^2 = 0; \quad (92)$$

$$\Delta_{-3} = \frac{1}{\sqrt{3!}} c_{\alpha\beta\gamma} \partial_{\Theta_\alpha} \partial_{\Theta_\beta} \partial_{\Theta_\gamma}, \quad (\Delta_{-3})^2 = 0. \quad (93)$$

The nilpotency of the operators Δ_{+1} and Δ_{-1} is a consequence of relations (85) and (87). The operator Δ_{+1} is proportional to the second term in a BRST charge

$$Q = \Theta^\alpha G_\alpha - \frac{1}{2} \Theta^\alpha \Theta^\beta c_{\alpha\beta}^\gamma \partial_{\Theta_\gamma},$$

where Θ^α and ∂_{Θ^α} represent the operators for ghosts and antighosts respectively. Q itself will be proportional to the operator Δ_{+1} if we take the representation S_α (89) for group generators G_α . The operator Δ_{-1} , related with the divergence of a vector field $\{\Theta_\alpha, A\}_1$

$$\partial_{\Theta_\alpha} \{\Theta_\alpha, A\}_1 = \partial_{\Theta_\alpha} S_\alpha A = -\sqrt{2} \Delta_{-1} A,$$

is proportional to the true Δ -operator for the bracket (83).

It is also interesting to reveal that these Δ -like operators together with the Casimir function Δ_{+3} (88) are closed into the finite-dimensional Lie superalgebra, in which the anticommuting relations between the quantities Δ_λ ($\lambda = -3, -1, +1, +3$) (88), (91)–(93) with the nonzero right-hand side are

$$\{\Delta_{-1}, \Delta_{+1}\} = Z, \quad (94)$$

$$\{\Delta_{-3}, \Delta_{+3}\} = N - 3Z, \quad (95)$$

where

$$N = -c^{\alpha\beta\gamma} c_{\alpha\beta\gamma}$$

is a number of values for the indices α, β, γ ($\alpha, \beta, \gamma = 1, \dots, N$) and

$$Z = D - K \quad (96)$$

is a central element of this superalgebra

$$[Z, \Delta_\lambda] = 0, \quad (\lambda = -3, -1, +1, +3). \quad (97)$$

In (96)

$$D = \Theta^\alpha \partial_{\Theta_\alpha} \quad (98)$$

is a "dilatation" operator for the Grassmann variables Θ_α , which distinguishes the Δ_λ -operators with respect to their uniformity degrees in Θ

$$[D, \Delta_\lambda] = \lambda \Delta_\lambda, \quad (\lambda = -3, -1, +1, +3) \quad (99)$$

and is in fact a representation for a ghost number operator, and the quantity K has the form

$$K = \frac{1}{2} \Theta^\alpha \Theta^\beta c_{\alpha\beta}^\lambda c_{\lambda\gamma\delta} \partial_{\Theta_\gamma} \partial_{\Theta_\delta}. \quad (100)$$

The operator Z is also a central element of the Lie superalgebra which contains both the operators

Δ_λ (88), (91)–(93), Z (96) and the operator D (98)

$$[Z, D] = 0. \quad (101)$$

We can add to this superalgebra the generators S_α (89) with the following commutation relations:

$$[S_\alpha, \Delta_\lambda] = 0, [S_\alpha, Z] = 0, [S_\alpha, D] = 0, \quad (102)$$

which indicate that both the Casimir function Δ_{+3} and the operators Δ_λ ($\lambda = -3, -1, +1$), Z and D are invariants of the Lie group with the generators S_α . In order to prove the permutation relations for the Lie superalgebra (88)–(102), we have to use relations (84)–(87). Note that the central element Z (96) coincides with the expression for a quadratic Casimir operator of the Lie algebra (90) for the generators S_α given in the representation (89)

$$S_\alpha S_\beta g^{\alpha\beta} = Z. \quad (103)$$

Thus, we see that both the even and odd linear Poisson brackets are internally inherent in the Lie group with the structure constants subjected to conditions (78) and (79). However, only for the linear odd Poisson bracket realized in terms of the Grassmann variables and only in the case when this bracket corresponds to the semi-simple Lie group, there exists the Lie superalgebra (88)–(102) for the Δ -like operators of this bracket.

Note that in the case of the degenerate Cartan-Killing metric tensor (80), relation (81) remains valid and we can construct only two Δ -like Grassmann-odd nilpotent operators: Δ_{-1} (92) and Δ_{-3} (93), which satisfy the trivial anticommuting relation

$$\{\Delta_{-1}, \Delta_{-3}\} = 0.$$

Note also that anticommuting relations

$$\{\Delta_{-1}^i, \Delta_{-1}^k\} = -2 c_{\alpha\beta}^{\{i} c_{\lambda\gamma}^{k\} \Theta_\delta \partial_{\Theta_\alpha} \partial_{\Theta_\beta} \partial_{\Theta_\gamma}$$

for the operators

$$\Delta_{-1}^i = \frac{1}{\sqrt{2}} \Theta_\gamma c_{\alpha\beta}^i \partial_{\Theta_\alpha} \partial_{\Theta_\beta}, \quad (\Delta_{-1})^2 = 0,$$

corresponding to the Lie algebras with structure constants $c_{\alpha\beta}^i$ ($i = 1, 2$), vanish provided that

$c_{\alpha\beta}^i$ satisfy compatibility conditions [28]

$$\sum_{(\alpha\beta\gamma)} c_{\alpha\beta}^{\{i} c_{\lambda\gamma}^{k\}} = 0,$$

where $\{ik\}$ denotes a symmetrization of the indices i and k .

The Lie superalgebra (88)–(102), naturally connected with the linear odd Poisson bracket (83), may be useful for the subsequent development of the Batalin-Vilkovisky formalism for the quantization of gauge theories. Let us note that this superalgebra can also be used in the theory of representations of the semi-simple Lie groups.

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